Derivation of Kepler's third law

We will derive Kepler's third law for the special case of a circular orbit. (Most planetary orbits are close to a circle.) First, we write Newton's second law of motion, $\Sigma F = ma$. For F we use the gravitational force (Eq. 5–4) between the Sun and a planet of mass m_1, and for a the centripetal acceleration, v^2/r. We assume the mass of the Sun, M_S, is much greater than the mass of its planets. Then

$$\Sigma F = ma$$

$$G\frac{m_1 M_S}{r_1^2} = m_1 \frac{v_1^2}{r_1}.$$

Here r_1 is the distance of one planet from the Sun, and v_1 is its average speed in orbit; M_S is the mass of the Sun, since it is the gravitational attraction of the Sun that keeps each planet in its orbit. The period T_1 of the planet is the time required for one complete orbit, a distance equal to the circumference of its orbit, $2\pi r_1$. Thus

$$v_1 = \frac{2\pi r_1}{T_1}.$$

We substitute this formula for v_1 into the equation above:

$$G\frac{m_1 M_S}{r_1^2} = m_1 \frac{4\pi^2 r_1}{T_1^2}.$$

We rearrange this to get

$$\frac{T_1^2}{r_1^3} = \frac{4\pi^2}{GM_S}. \qquad \textbf{(5–6a)}$$

We derived this for planet 1 (say, Mars). The same derivation would apply for a second planet (say, Saturn) orbiting the Sun,

$$\frac{T_2^2}{r_2^3} = \frac{4\pi^2}{GM_S},$$

where T_2 and r_2 are the period and orbit radius, respectively, for the second planet. Since the right sides of the two previous equations are equal, we have $T_1^2/r_1^3 = T_2^2/r_2^3$ or, rearranging,

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$$\left(\frac{T_1}{T_2}\right)^2 = \left(\frac{r_1}{r_2}\right)^3, \qquad \textbf{(5–6b)}$$

which is Kepler's third law.

The derivations of Eqs. 5–6a and 5–6b (Kepler's third law) compared two planets revolving around the Sun; but they are general enough to be applied to other systems. For example, we could apply Eq. 5–6a to our Moon revolving around Earth (then M_S would be M_E, the mass of the Earth). Or we could apply Eq. 5–6b to compare two moons revolving around Jupiter. But Kepler's third law applies only to objects orbiting the same attracting center. Do not use Eq. 5–6b to compare, say, the Moon's orbit around the Earth to the orbit of Mars around the Sun because they depend on different attracting centers.

CAUTION

Compare orbits of objects only around the same center

In the following Examples, we assume the orbits are circles, although it is not quite true in general.

EXAMPLE 5–15 **Where is Mars?** Mars' period (its "year") was noted by Kepler to be about 687 days (Earth days), which is (687 d/365 d) = 1.88 yr. Determine the distance of Mars from the Sun using the Earth as a reference.

APPROACH We know the periods of Earth and Mars, and the distance from the Sun to Earth. We can use Kepler's third law to obtain the distance from the Sun to Mars.